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SYNCHRONIZATION OF WEAKLY COUPLED RELAXATION OSCILLATORS

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Synchronization of weakly coupled relaxation oscillators

by

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ABSTRACT

Existence and approximation problems for periodic solutions of a system of weakly coupled relaxation oscillators are reduced to an algebraic problem. The starting point is a theorem by Mishchenko on periodic solutions of certain singularly perturbed ordinary differential equations. In this report we prove some theorems stated in MC report TW 178/78.

KEY WORDS & PHRASES: *relaxation oscillators, singular perturbations, periodic solutions of differential equations.*

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1. INTRODUCTION

In this report we treat the following system of ordinary differential equations:

$$\text{(1.1)} \qquad \begin{aligned} \varepsilon\dot{u}_i &= v_i - F(u_i) \\ \dot{v}_i &= -u_i + \delta h_i(u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n) \end{aligned} \qquad (i = 1,2,\ldots,n),$$

where $u_i(t)$ and $v_i(t)$ are real valued functions of the time $t \in [0,\infty)$, $\dot{} = d/dt$, and where ε and δ are small parameters: $\varepsilon \in (0,\varepsilon_0)$, $\delta \in [0,\delta_0)$. The real valued functions F and h_i have continuous (mixed) derivatives of all orders.

In section 2 we briefly discuss the behaviour of the solutions of (1.1) for $\delta = 0$. In that case (1.1) reduces to n identical equations of the form

$$\text{(1.2)} \qquad \begin{aligned} \varepsilon\dot{x} &= y - F(x) \\ \dot{y} &= -x \end{aligned}$$

The function F satisfies conditions ensuring that (1.2) has an asymptotically stable periodic solution for $\varepsilon > 0$. For small values of ε periodic solutions of (1.2) are called relaxation oscillations.

In section 3 equation (1.1) is investigated; this equation may be considered as a system of n coupled relaxation oscillators in which the functions h_i represent the coupling.

Equation (1.1) contains two small parameters. First we investigate the *singular perturbation problem* $\varepsilon \to 0$, δ fixed. In that case the solutions of (1.1) will tend (in a certain sense) to so called singular or discontinuous solutions. Suppose that (1.1) has a periodic singular solution satisfying certain stability requirements. Then by a theorem of MISHCHENKO [7] a positive $\bar{\varepsilon}$ exists such that (1.1) has a periodic solution for $0 < \varepsilon \leq \bar{\varepsilon}$.

Thus we may restrict our attention to singular solutions of (1.1). It will be seen that an invariant torus exists with respect to these solutions. This torus is the n-fold cartesian product of the singular limit cycle of

(1.2). We define a parametrisation of this torus which satisfies a *regular perturbation problem* for $\delta \to 0$.

"Stable" periodic singular solutions of (1.1) are detected with a Poincaré map which can be approximated for small values of δ.

Finally we prove a theorem reducing problems of existence and approximation of periodic solutions of (1.1) to an algebraic problem.

The motivation for this report largely came from a paper by WINFREE [13] on hypothetical oscillators. These oscillators were substantialized by Grasman in a study on two coupled relaxation oscillators [11; vol. 2, Ch. I]. In GRASMAN & JANSEN [2] the theory of weakly coupled relaxation oscillators is further developed and many special cases of (1.1) are studied as analogues of oscillating systems in biology.

2. ONE RELAXATION OSCILLATOR

In this section we shall briefly discuss the behaviour of one isolated relaxation oscillator. This will serve as an introduction to section 3, where the behaviour of coupled oscillators is discussed.

One oscillator shall be described by a system of two first order equations for the real-valued functions $x(t)$ and $y(t)$ (t real):

$$\varepsilon\dot{x} = y - F(x),$$
(2.1)
$$\dot{y} = -x,$$

where ε is a small positive parameter and where $\dot{}$ = d/dt. With respect to $F\colon \mathbb{R} \to \mathbb{R}$ we assume that

$$F(x) = -F(-x), \qquad (x \in \mathbb{R}),$$
(2.2.a)
$$F \in C^1(\mathbb{R}),$$
$$F(x) \to +\infty, \qquad (x \to +\infty).$$

Moreover, a positive number m should exist such that the derivative of F, denoted by f, satisfies

$$f(x) < 0, \qquad x \in (0,m),$$

(2.2.b)

$$f(x) > 0, \qquad x \quad (m,\infty).$$

THEOREM 2.1. *Under conditions* (2.2.a) *and* (2.2.b) *equation* (2.1) *has a unique periodic solution which is asymptotically stable.*

PROOF. See HALE [3].

For weaker conditions on the function F see LASALLE [6].

In the rest of this report, we shall assume that the following two extra conditions are satisfied:

$$f'(m) \neq 0,$$

(2.2.c)

$$F \in C^{\infty}(\mathbb{R}).$$

The classical example of an equation of type (2.1) satisfying conditions (2.2) is the Van der Pol equation in which $F(x) = x^3/3 - x$ (see for instance [2] or [3]).

In order to approximate the solutions of (2.1) we introduce the *reduced system:*

$$y = F(x), \tag{2.3.a}$$

$$\dot{y} = -x, \tag{2.3.b}$$

which is obatined from (2.1) by substituting $\varepsilon = 0$. It follows from (2.3.a) and (2.3.b) that

$$f(x)\dot{x} + x = 0, \qquad (f(x) \neq 0). \tag{2.3.c}$$

We also introduce the *fast equation*

$$\varepsilon\dot{x} = y - F(x), \qquad (y \text{ constant}), \tag{2.4}$$

in which the constant y is considered as parameter.

With the aid of (2.3) and (2.4) we can define the *singular* or *discontinuous* solution of (2.1), which approximates the solution of (2.1) in a sense that will be precised later. Let the singular solution start at time zero in the point $(\tilde{x},\tilde{y})$. If $\tilde{y} \neq F(\tilde{x})$ the singular solution makes an instantaneous jump along the trajectory of the fast equation until a stable equilibrium $(x_s,\tilde{y})$, with $F(x_s) = \tilde{y}$ and $f(x_s) > 0$, is reached. Such a point will be called a *landing point*. From then on the singular solution satisfies the reduced equation (2.3). This part of the singular solution is called *regular*. Along the regular part the absolute value of x decreases until x attains a local extremum of $F(x)$ ($x = \pm m$, $f(x) = 0$). At that moment $y = F(x)$ ceases to be a stable equilibrium of (2.4). The point where this happens is called a *leaving point*. It follows from the conditions (2.2) that equation (2.4) has only one trajectory departing from a leaving point. The singular solution makes an instantaneous jump along this trajectory until a new landing point is reached, after which the reduced equation is satisfied again. Thus the singular solution is described alternately by instantaneous jumps along trajectories of the fast equation and by regular parts satisfying the reduced equation.

THEOREM 2.2. *Let* ξ_ε *and* ξ_0 *denote the trajectories of a solution respectively a singular solution of* (2.1) *that start in the same point of* $\mathbb{R}^2$. *Then* $\xi_\varepsilon \to \xi_0$ *for* $\varepsilon \to 0$. *The convergence is uniform on parts of* ξ_ε *corresponding to bounded time intervals.*

PROOF. See MISHCHENKO & ROZOV [9].

The approximation near the regular parts and the jump parts has been studied by TIKHONOV [12] (see also HOPPENSTEADT [5]). The approximation near the leaving points has been studied by MISHCHENKO and PONTRYAGIN [8], [10]. A general survey of the theory is given by MISHCHENKO and ROZOV [9].

From the conditions (2.2) it follows (as illustrated in figure 2.1) that equation (2.1) has a unique periodic singular solution, which we shall indicate by $(x^0(t), y^0(t))$. The closed trajectory X_0 of this solution proceeds along ABCD as sketched in figure 2.1. The arcs AB and CD represent the regular parts; BC and DA represent the jumps.

Let the periodic singular solution $(x^0(t), y^0(t))$ start at $t = 0$ in the point A (figure 2.1). In order to calculate the period T_0 of this solution we remark that on the regular part AB, where x decreases from, say, M to m, the time t is given according to (2.3.c) by

$$t = - \int_M^x \frac{f(\xi)}{\xi} \, d\xi. \tag{2.5}$$

This means that the solution runs through AB in the time

$$\int_m^M \frac{f(x)}{x} \, dx.$$

Because of the symmetry of F (2.2) the part CD takes the same time, whereas BC and DA represent instantaneous jumps. Consequently the period of the singular solution is given by

$$T_0 = 2 \int_m^M \frac{f(x)}{x} \, dx. \tag{2.6}$$

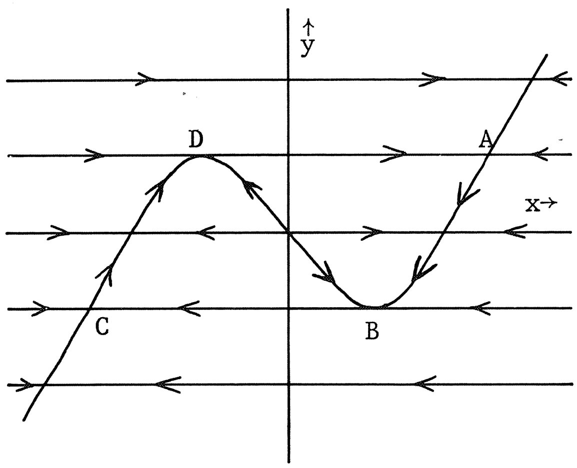

Fig. 2.1. State space of singular solutions

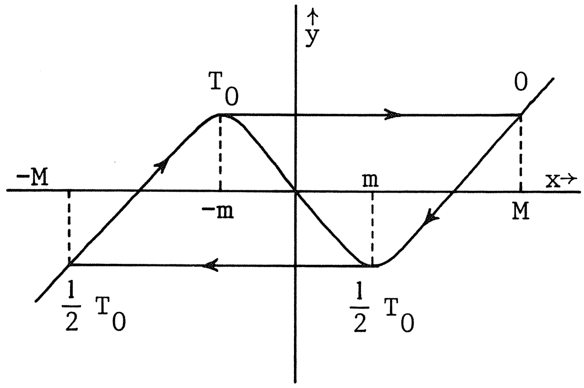

Fig. 2.2. The periodic singular solution

In figure 2.1 the state space of the singular solutions of (2.1) is drawn. The arrows indicate the change of state; horizontal arrows indicate instantaneous changes. The closed trajectory X_0 = ABCD represents the periodic singular solution. On this trajectory the state of the singular solution can be represented by one real variable, called the phase, which coincides for isolated oscillators with time. The period of the singular solution is denoted by T_0. In figure 2.2 the closed trajectory X_0 is drawn. At the points A, B, C and D the phases and the values of x are indicated. In the pictures $F(x) = x^3/3 - x$ (Van der Pol equation).

We shall investigate the regularity properties of $x^0(t)$. For definiteness we consider $x^0(t)$ to be continuous from the right at the jump points $\frac{1}{2}jT_0$ (j integer). From the smoothness of f (see (2.2.c)) it follows via (2.3.c) that on the regular parts x^0 has continuous derivatives of all orders. These derivatives are bounded except in left-neighbourhoods of the jump points. We will consider the behaviour of $x^0(t)$ in a left neighbourhood of $\frac{1}{2}T_0$. The behaviour near the other jumps follows by the symmetry of F (see (2.2)). When $x^0 - m \downarrow 0$ we have according to (2.5)

$$t = -\int_M^{x^0} \frac{f(\xi)}{\xi}\,d\xi = \frac{1}{2}T_0 - \int_0^{x^0-m} \frac{f(m+r)}{m+r}\,dr,$$

in which f can be developed in a Taylor-series around m (see (2.2)):

$$\text{(2.6.a)} \qquad t - \frac{1}{2}T_0 = -\int_0^{x^0-m} \frac{rf'(m)+\ldots}{m+r}\,dr = -\frac{f'(m)}{2m}(x^0-m)^2 + \ldots\,.$$

It follows that x^0-m can be developed in powers of $\sqrt{\frac{1}{2}T_0-t}$:

$$\text{(2.6.b)} \qquad x^0 - m = \sqrt{2m/f'(m)}\sqrt{\tfrac{1}{2}T_0-t} + \ldots, \qquad (\tfrac{1}{2}T_0-t \downarrow 0).$$

For later reference we summarize some direct conclusions from the foregoing:

LEMMA 2.1. *The component* $x^0(t)$ *of the periodic singular solution of* (2.1) *is piecewise Hölder continuous with exponent* $\frac{1}{2}$ *and with jumps at* $\frac{1}{2}jT_0$ *(j integer). Moreover* $x^0(t)$ *is* C^∞ *and uniformly Lipschitz continuous on compact sets not containing jump points.*

The following theorem establishes a relation between the periodic solution of (2.1) for some $\varepsilon > 0$, and the singular solution $(x^0(t), y^0(t))$. For the proof the reader is referred to MISHCHENKO and PONTRYAGIN [7], [8], [10]:

THEOREM 2.3. *Let* F *satisfy* (2.2). *Then the period* T_ε *of the asymptotically stable periodic solution of* (2.1) *satisfies:* $T_\varepsilon = T_0 + O(\varepsilon^{2/3})$ $(\varepsilon \to 0)$. *The limit cycle* X_ε *of* (2.1) *satisfies:* $X_\varepsilon \to X_0$ *uniformly for* $\varepsilon \to 0$.

3. SYSTEMS OF RELAXATION OSCILLATORS

In this section we shall investigate a finite system of coupled relaxation oscillators described by the following differential equations:

$$\begin{aligned} &(3.1.a) \qquad \varepsilon\dot{u}_i = v_i - F(u_i) \\ & \qquad\qquad\qquad\qquad\qquad\qquad (i = 1,2,\ldots,n), \\ &(3.1.b) \qquad \dot{v}_i = -u_i + \delta h_i(u,v) \end{aligned}$$

where n denotes the number of oscillators and where $u = (u_1,u_2,\ldots,u_n)$, $v = (v_1,v_2,\ldots,v_n)$. The parameters ε and δ are assumed to be small and positive. The function F satisfies the conditions (2.2). The functions h_i represent the coupling between the oscillators. We shall assume that these functions have continuous derivatives of any order:

$$(3.1.c) \qquad h_i \in C^\infty(\mathbb{R}^{2n}), \qquad (i = 1,2,\ldots,n).$$

The main result of this section (theorem 3.3) can also be proved when the latter condition is weakened.

We will investigate the behaviour of the solutions of (3.1) for ε tending to zero. Just as in the previous section we introduce the *reduced system*.

$$\begin{aligned} &(3.2.a) \qquad v_i = F(u_i) \\ & \qquad\qquad\qquad\qquad\qquad\qquad (i = 1,2,\ldots,n), \\ &(3.2.b) \qquad \dot{v}_i = -u_i + \delta h_i(u,v) \end{aligned}$$

Combination of these equations yields a reduced equation for u:

$$(3.2.c) \qquad f(u_i)\dot{u}_i = -(1 - \delta q_i)u_i + \delta h_i(u,\bar{F}(u)),$$

where $\bar{F}(u) = (F(u_1),F(u_2),\ldots,F(u_n))$ and where $f = F'$. We also introduce the *fast system*

$$(3.3) \qquad \varepsilon\dot{u}_i = v_i - F(u_i), \qquad (v_i \text{ constant}),$$

in which the constants v_i are considered as parameters.

The *singular solution* of (3.1) is defined as follows: When (u,v) is not a stable equilibrium point of the fast system an instantaneous jump is made along a trajectory of the fast equation until a stable equilibrium of this equation is reached (a landing point), where $v_i = F(u_i), f(u_i) > 0$

for $i = 1,2,\dots,n$ (see figure 2.1). Afterwards the singular solution satisfies the reduced system until one or more of the variables $u_1,u_2,\dots,u_n$ reaches a local extremum of F (i.e. a zero, $\pm m$, of f). At that point (a leaving point) the unique solution of the reduced equation (3.2.c) cannot be further extended. The singular solution then makes an instantaneous jump along the uniqe trajectory of the fast equation departing from the leaving point until a new landing point is reached. After the jump the singular solution is described again by the reduced system.

The singular solution approximates the true solution of (3.1) in the sense indicated in theorem 2.2:

THEOREM 3.1. *Let ζ_ε and ζ_0 denote the trajectories of a solution respectively a singular solution of equation* (3.1) *that start in the same point of $\mathbb{R}^{2n}$. Then $\zeta_\varepsilon \to \zeta_0$ for $\varepsilon \to 0$. The convergence is uniform on parts of ζ_ε corresponding to bounded time intervals.*

PROOF. See MISHCHENKO & ROZOV [9].

It can moreover be shown that system (3.1) has a periodic solution if it has a periodic singular solution satisfying certain conditions. We have to introduce some concepts before formulating a theorem of this kind.

Let the regular parts AB and CD of the closed trajectory X_0 of the periodic singular solution of one oscillator (2.1), be indicated by Ω_0. Then we may define the following n-dimensional torus-like surface in the space $\mathbb{R}^{2n}$

$$(3.4) \qquad \Omega_0^n := \{(u,v) \in \mathbb{R}^{2n} \mid (u_i,v_i) \in \Omega_0 \text{ for } i = 1,2,\dots,n\}.$$

This means that $(u,v) \in \Omega_0^n$ when each oscillator (u_i,v_i) lies on the regular trajectory Ω_0 of the singular solution of one isolated oscillator. We shall show that for δ sufficiently small a singular solution $(u(t), v(t))$ will remain in the set Ω_0^n, once it has arrived there:

LEMMA 3.1. *When δ is sufficiently small but finite, the set Ω_0^n is invariant with respect to singular solutions of* (3.1).

PROOF. Let $(u,v) \in \Omega_0^n$. We first consider the case when $f(u_i) \neq 0$ $(i = 1,2,\dots,n)$. This means that $f(u_i) > 0$ and that $|u_i| > m$ (figure 2.1). Moreover (u,v) is bounded. It follows from (3.2) that for δ sufficiently small

$$\text{(3.5.a)} \qquad \left.\begin{aligned} v_i &= F(u_i) \\ \operatorname{sign}(\dot{u}_i) &= -\operatorname{sign}(u_i) \end{aligned}\right\} \qquad (i = 1,2,\dots,n).$$

From (2.3) we obtain for the isolated oscillator

$$\text{(3.5.b)} \qquad \begin{aligned} y &= F(x) \\ \operatorname{sign}(\dot{x}) &= -\operatorname{sign}(x). \end{aligned}$$

This implies that the coupled oscillators run through parts AB and CD in the same direction as the uncoupled oscillator. In the leaving points, where at least one of the functions $f(u_i)$ is zero, the system makes an instantaneous jump to a new stable equilibrium of (3.3). This equilibrium is the same for coupled and uncoupled oscillators. Consequently, Ω_0^n is invariant. □

For a concise formulation of a theorem on periodic solutions of (3.1) we will need two definitions:

DEFINITION 3.1. Let $n \geq 2$ and let W be a smooth $n-1$ dimensional surface lying in the n-dimensional surface Ω_0^n. Moreover, let W be nowhere tangent to the trajectories of the singular solutions of (3.1). Let w be a point of W and let a singular solution start in w. Then it may happen that this singular solution will return in W. If so, denote the point of first return by $P(w)$. In this way a mapping, P, is defined from a part of W into W. This mapping is called the *Poincaré map* of W produced by the singular solution.

The Poincaré map is commonly used to investigate the stability of solutions which are known to be periodic (see for instance HIRSCH & SMALE [4]). We shall use this mapping also to detect periodic solutions. It is clear that there exists a periodic singular solution if P has a *fixed point*, i.e. a point $w \in W$ such that $P(w) = w$. The closed trajectory of such a periodic solution will be indicated by $Z_0 \subset \mathbb{R}^{2n}$, the period by P_0.

DEFINITION 3.2. A periodic singular solution of (3.1) will be called C-*stable* if a surface W exists as described in definition 3.1, such that the corresponding Poincaré map, P, is *contracting* at the intersection of W and Z_0.

The following theorem, proved by MISHCHENKO [7], permits us to fix our attention to C-stable periodic singular solutions of equation (3.1):

THEOREM 3.2. *Let* δ *be such that equation* (3.1) *has a* C-*stable periodic singular solution with trajectory* Z_0 *and period* P_0. *Let only one of its components* $u_i(t)$ $(i = 1,2,\ldots,n)$ *be discontinuous at a time. Then a positive function* $\bar{\varepsilon}(\delta)$ *exists such that for* $0 < \varepsilon \le \bar{\varepsilon}(\delta)$, *equation* (3.1) *has a periodic solution with period* P_ε *and trajectory* $Z_\varepsilon \subset \mathbb{R}^{2n}$ *satisfying:*
(i) $Z_\varepsilon \to Z_0$ *uniformly for* $\varepsilon \to 0$, *and*
(ii) $P_\varepsilon = P_0 + O(\varepsilon^{2/3})$.

In the sequel we shall only consider singular solutions of (3.1) lying in Ω_0^n. The differential equation of such a solution will take on a particularly simple form when transformed by a suitable parametrisation of Ω_0^n. This parametrisation is given by the *phase map*:

$$\text{(3.6.a)} \qquad \Phi\colon (\phi_1,\ldots,\phi_n) \mapsto (x^0(\phi_1),\ldots,x^0(\phi_n),\ y^0(\phi_1),\ldots,y^0(\phi_n)).$$

Since x^0 and y^0 are T_0-periodic this map, which is surjective, can be made injective by identification of any two points ϕ and ψ with $\phi_i = \psi_i$ (modulo T_0) $(i = 1,2,\ldots,n)$. Consequently the map Φ will be defined on the n-dimensional *torus* T^n, the n-fold cartesian product of the circle $T := \mathbb{R}/T_0$:

$$\text{(3.6.b)} \qquad \Phi\colon T^n \to \Omega_0^n \ .$$

when so defined Φ is injective and (by lemma 2.1) diffeomorphic except at the jump planes $\phi_i = \frac{1}{2} j_i T_0$ (j integer). The point $\phi_i \in T$ will be called the *phase* of the i-th oscillator.

LEMMA 3.2. *A singular solution* $(u(t),v(t))$ *of equation* (3.1) *lying in* Ω_0^n *can be represented as*

(3.7.a) $$(u(t),v(t)) = \Phi(\phi(t)),$$

where the components ϕ_i *of the function* ϕ: $[0,\infty) \to T^n$ *satisfy the phase equation:*

(3.7.b) $$\dot{\phi}_i = 1 + \delta k_i(\phi) \qquad (i = 1,2,\ldots,n),$$

in which k_i: $T^n \to \mathbb{R}$ *is defined by*

(3.7.c) $$k_i(\phi) = -h_i(\Phi(\phi))/x^0(\phi_i).$$

PROOF. Equation (3.7.b) describes the singular solution both on the regular parts and on the singular parts. On the regular parts (3.7.b) is obtained by substituting (3.7.a) into (3.2.c), making use of (2.3.c). At the end of a regular part the system makes an instantaneous jump to a new stable equilibrium of (3.3). Since the oscillators are uncoupled in (3.3), the behaviour in the jumps is also correctly described by (3.7.b). □

Note that the functions k_i are bounded since x^0 is bounded away from zero.

We shall now approximate the solution of (3.7.b) in order to investigate the mapping P. Since δ is a small parameter it is natural to try to solve (3.7.b) by iteration. Let $\phi(0) = \alpha \in T^n$. Then the first and second iterates are

(3.8.a) $$\phi_i^{(0)}(t) = \alpha_i + t$$

(3.8.b) $$\phi_i^{(1)}(t) = \alpha_i + t + \delta \int_0^t k_i(\alpha_1+\tau,\ldots,\alpha_n+\tau)dt.$$

LEMMA 3.3. *Equation* (3.7.b) *with initial condition* $\phi(0) = \alpha$ *has a unique solution* $\phi(t)$ *satisfying*

(3.9.a) $$\phi_i(t) = \phi_i^{(1)}(t) + O(\delta^{3/2}), \qquad \text{(t bounded)}.$$

PROOF. Lemma 3.2 shows that there is a 1-1 correspondence between singular solutions of (3.1) lying in Ω_0^n and solutions of (3.7.b). The singular solution of (3.1) with initial value $\Phi(\alpha) \in \Omega_0^n$ exists and is unique by definition. This solution remains in Ω_0^n by lemma 3.1. It follows that (3.7.b) has a unique solution.

The correctness of (3.9) remains to be proved. We start with an investigation of the regularity of the function $k(\phi)$ given in (3.7.c). It was shown in lemma 2.1 that x^0 is piecewise Hölder continuous with exponent $\frac{1}{2}$ and with finite jumps at the points $\frac{1}{2}\, jT_0$ (j integer). Since h and F are C^∞-functions and since x^0 is bounded away from zero it follows from (3.7.c) that $k(\phi)$ is piecewise Hölder continuous with exponent $\frac{1}{2}$ and with finite jumps in the planes $\phi_i = \frac{1}{2}\, j_i T_0$ (j_i integer).

This property of $k(\phi)$ may be used to estimate $\phi(t) - \phi^{(1)}(t)$ which, according to (3.7.b) and (3.8), can be written as

$$\phi(t) - \phi^{(1)}(t) = \delta \int_0^t [k(\phi(t)) - k(\phi^0(t))]dt. \tag{3.9.b}$$

It follows from (3.7) and (3.8) that $\phi(t) - \phi^0(t) = O(\delta)$ (t bounded). The space T^n is partitioned by the planes $\phi_i = \frac{1}{2}\, j_i T_0$ (j_i integer) into n-dimensional cubes where $k(\phi)$ is Hölder continuous. When $\phi(t)$ and $\phi^0(t)$ are in the same cube one has

$$|k(\phi(t)) - k(\phi^{(0)}(t))| \le H|\phi(t) - \phi^{(0)}(t)|^{\frac{1}{2}} = O(\delta^{\frac{1}{2}}),$$

in which H is the Hölder constant of k. This gives an $O(\delta^{3/2})$ contribution to (3.9.b). However $\phi(t)$ and $\phi^0(t)$ may be located in different cubes, in which case $k(\phi(t)) - k(\phi^{(0)}(t)) = O(1)$. It follows from $\phi^0(t) = \alpha + t(1,1,\ldots,1)$, that $\phi^0(t)$ is never tangent to the jump planes. This implies that $\phi^0(t)$ will remain in an $O(\delta)$ neighbourhood of the jump planes only during $O(\delta)$ time intervals. Using $\phi(t) - \phi^{(0)}(t) = O(\delta)$ we obtain that $\phi(t)$ and $\phi^0(t)$ can only stay in different cubes for $O(\delta)$ time intervals. These give an $O(\delta^2)$ contribution to (3.9.b). The correctness of (3.9.a) follows immediately. □

We shall frequently need a special condition on the initial value $\phi(0) = \alpha$. Therefore we state

DEFINITION 3.3. A point $\alpha \in T^n$ will be called *regular* if the functions $x^0(\alpha_i + t)$ $(i = 1,2,\ldots,n)$ are continuous in $t = 0$ and if they become discontinuous one at a time.

Since x^0 is discontinuous at $\frac{1}{2} jT_0$ (j integer) the regularity condition implies that $\alpha_i \neq \frac{1}{2} jT_0$ and that no two of the phases α_i are equal or complementary.

The Poincaré map P of definition 3.1 can be investigated with the aid of approximation (3.9.a). Let V be an n-1 dimensional plane in T^n perpendicular to the vector $e := (1,1,\ldots,1) \in T^n$. Let α be a regular point in V, let $U \subset V$ be a neighbourhood of α, and let $W := \Phi(U)$. Since Φ is diffeomorphic outside the jumps, the n-1 dimensional surface $W \subset \Omega_0^n$ satisfies the conditions of definition 3.1. The Poincaré map P: $W \to W$ may be derived from the map P^*: $U \to U$ produced by following the trajectories of (3.7.b) (see figure 3.1).

LEMMA 3.4. *Let* $\tilde{\alpha} \in T^n$ *be regular and let* U, W *and* P^* *be defined as above. Then the Poincaré map* P: $W \to W$ *is given by*

$$(3.10) \qquad P = \Phi P^* \Phi^{-1}.$$

The point $\Phi(\tilde{\alpha})$ *is a contracting fixed point of* P *if* $P^*(\tilde{\alpha}) = \tilde{\alpha}$ *and if the eigenvalues of the derivative of* P^* *in* $\tilde{\alpha}$ *have absolute values less than one.*

PROOF. Representation (3.10) follows from lemma 3.2. The rest of the lemma is proved by defining a distance in $W \subset \Omega_0^n$ as the corresponding euclidean distance in $U \subset T^n$, produced by the map Φ^{-1}. □

We shall find an approximation of P^* for small values of δ. Substituting $t = T_0 + s$ with $s = O(\delta)$ in (3.8.b) we obtain for $\phi(T_0 + s) \in T^n$:

$$(3.11.a) \qquad \phi(T_0 + s) = \alpha + se + \delta G(\alpha) + O(\delta^{3/2}),$$

where G: $T^n \to T^n$ is given by its components

$$(3.11.b) \qquad G_i(\alpha) = \int_0^{T_0} k_i(\alpha_1 + \tau, \ldots, \alpha_n + \tau)dt.$$

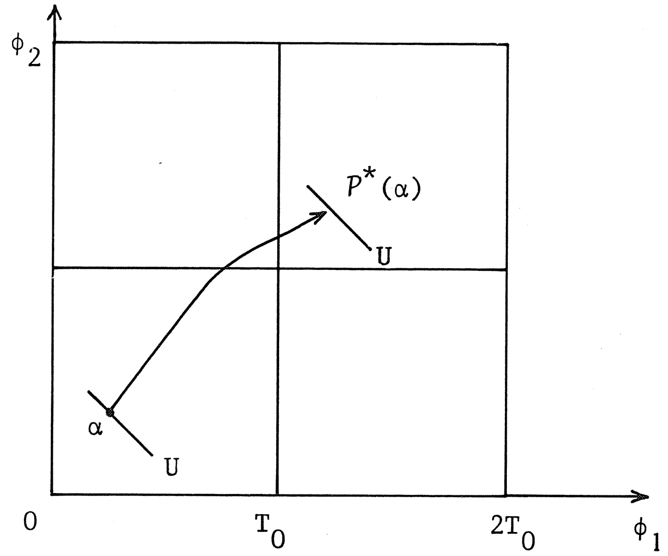

Fig. 3.1. The map P^* for two oscillators, produced by the trajectories of (3.7.b). The Poincaré map is given by $P = \Phi P^* \Phi^{-1}$. For clarity of the drawing the torus $T^2 = (\mathbb{R}/T_0)^2$ has been unrolled.

The function $G(\alpha)$ gives the *phase shift* caused by the coupling. Since k_i is defined on T^n (lemma 3.2) the function $G(\alpha)$ depends only on the phase differences $\alpha_i - \alpha_j$:

$$(3.11.c) \qquad G(\alpha + ce) = G(\alpha), \qquad (c \in \mathbb{R}),$$

We shall fix $s(\alpha)$ so that $\phi(T_0 + s(\alpha)) \in V$. This yields

$$(3.12) \qquad s(\alpha) = -\frac{\delta}{n} \sum G_i(\alpha) + O(\delta^{3/2}).$$

It follows that $\alpha \in V$ is mapped by P^* on

$$(3.13.a) \qquad \phi(T_0 + s(\alpha)) = \alpha + \delta Q(\alpha) + O(\delta^{3/2}),$$

where $Q\colon T^n \to T^n$ is defined as

$$(3.13.b) \qquad Q(\alpha) = G(\alpha) - (\frac{1}{n} \sum G_i(\alpha))e.$$

Note that

$$(3.13.c) \qquad Q(\alpha) \perp e,$$

which implies that for $\alpha \in V$

$$(3.13.d) \qquad \alpha + \delta Q(\alpha) \in V.$$

It follows from (3.11.c) and (3.13.b) that

$$(3.13.e) \qquad Q(\alpha + ce) = Q(\alpha) \qquad (c \in \mathbb{R})$$

We may summarize tha above calculations by stating:

LEMMA 3.5. *The asymptotic behaviour of the map* P^* *for* $\delta \to 0$ *is given by*

$$(3.14) \qquad P^*(\alpha) = \alpha_u + \delta Q_u(\alpha) + O(\delta^{3/2})$$

where α_u *and* $Q_u(\alpha)$ *denote the restrictions of* α *and* $Q(\alpha)$ *to the plane* U.

Using this lemma we can prove:

LEMMA 3.6. *Suppose that*

(i) $\tilde{\alpha} \in T^n$ *is regular (regularity condition),*

(ii) *all eigenvalues, except one, of the derivative of* $Q(\alpha)$ *in* $\tilde{\alpha}$ *have negative real parts (stability condition).*

(iii) $Q(\tilde{\alpha}) = 0$ *(synchronization condition).*

Then a point $\tilde{\beta} = \tilde{\alpha} + O(\delta)$ *exists in* T^n *such that* $P^*(\tilde{\beta}) = \tilde{\beta}$, *and such that the eigenvalues of the derivative of* P^* *in* $\tilde{\beta}$ *have absolute values less than one.*

PROOF. We shall prove the lemma in three steps:

(1) The mapping $P^*(\alpha)$ is C^∞ with respect to α and δ.

(2) Consequently $\alpha_u + \delta Q_u(\alpha)$ is an $O(\delta^2)$ approximation of $P^*(\alpha)$ which remains correct when differentiated with respect to .

(3) The lemma follows.

Proof of (1): Due to the fact that the left hand side of (3.7.b) is not Lipschitz continuous it is not immediately clear that P^* is C^∞ in α and δ. This problem can be overcome by a good choice of independent variables. Since $\phi(t) - \phi^{(0)}(t) = \phi(t) - (\tilde{\alpha} + te) = O(\delta)$, we know that $\phi(t)$ will remain in an $O(\delta)$ tube, N_δ, around the trajectory of $\tilde{\alpha} + te$. By the regularity of $\tilde{\alpha}$ this trajectory will not cross points where $k(\phi)$ is discontinuous in more than one variable. It follows that δ can be chosen so small that N_δ does not contain such points either. This implies that we know the order in which the variables $\phi_1, \phi_2, \ldots, \phi_n$ will cause the discontinuities of $k(\phi(t))$; this order will be indicated by $\phi_{i_1}, \phi_{i_2}, \ldots$. The corresponding planes of intersection of N with the discontinuity-planes of $k(\phi)$ will be denoted by $U_1, U_2, \ldots$ (see figure 3.2).

The map P^*: $U \to U$ is the composition of the mappings $U \to U_1$, $U_1 \to U_2, \ldots$ $\ldots, U_\ell \to U$, produced by the trajectories of (3.7.b) (the plane U_ℓ is the last of the planes $U_1, U_2 \ldots$ that is reached before U is reached). We shall show that these mappings are C^∞ with respect to δ and the initial conditions. This implies that P^* is C^∞ with respect to δ and α.

In the part of N bounded by U and U_1 we shall take ϕ_{i_1} as independent variable, so that equation (3.7.b) gets the form:

$$(3.7.b') \qquad \frac{d\phi_i}{d\phi_{i_1}} = \frac{1 + \delta k_i(\phi)}{1+\delta k_{i_1}(\phi)} \qquad (i \neq i_1).$$

For δ sufficiently small this equation has bounded derivatives of any order with respect to δ and ϕ_i $(i \neq i_1)$; moreover it is continuous with respect to ϕ_{i_1}.

It follows from [1; Ch. I, thrm 7.5] that the mapping $U \to U_1$ produced by the trajectories of (3.7.b) is C^∞ with respect to δ and the initial condition. The other mappings are treated in the same way; for the map $U_\ell \to U$ we take $\Sigma\phi_i$ as independent variable.

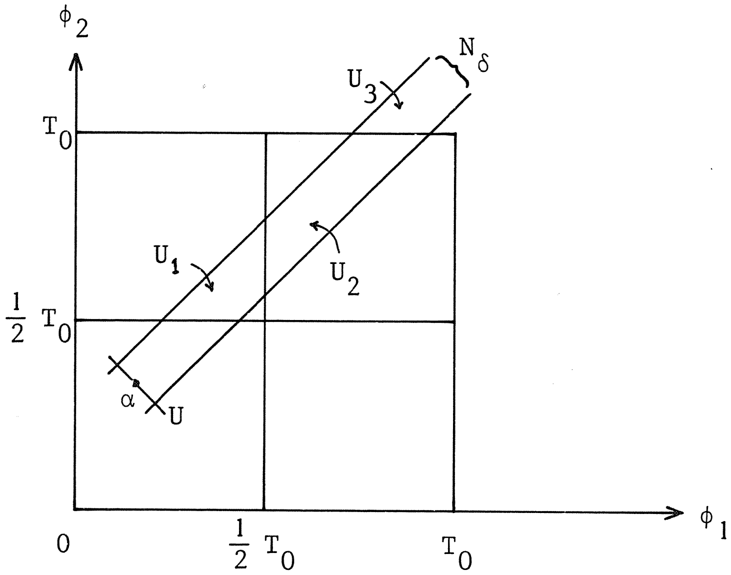

Fig. 3.2. The tube N_δ and the planes U_1, U_2, etc., for two oscillators. For clarity of the drawing the torus T^2 has been unrolled.

Proof of (2): It follows from (1) that P^* and its derivatives with respect to α (denoted by $DP^*, D^2P^*\ldots$) can be developed in a Taylor series around $\delta = 0$:

$$P^*(\alpha) = A(\alpha) + \delta B(\alpha) + O(\delta^2),$$
$$DP^*(\alpha) = DA(\alpha) + \delta DB(\alpha) + O(\delta^2),$$
$$\vdots$$

Using lemma 3.5 we obtain

$$P^*(\alpha) = \alpha_u + \delta Q_u(\alpha) + O(\delta^2),$$

$$DP^*(\alpha) = D\alpha_u + \delta DQ_u(\alpha) + O(\delta^2), \tag{3.14'}$$

$$D^2P^*(\alpha) = O(\delta).$$

Note that (3.14') gives a better estimate of P^* than (3.14).

Proof of (3): According to (3.13.e) the derivative $D(\alpha+\delta Q(\alpha))$ has an eigenvalue 1 corresponding to the eigenvector e. According to (3.13.d) $\alpha+\delta Q(\alpha)$ maps U into U. This implies that $D(\alpha_u+\delta Q_u(\alpha))$ is contracting if the other eigenvalues of $D(\alpha+\delta Q(\alpha))$ are lying strictly within the unit circle. It now follows from the conditions of the lemma that $\alpha_u+\delta Q_u(\alpha)$ is contracting in $\tilde{\alpha}$. It remains to be shown that a point $\tilde{\beta}$ exists, close to $\tilde{\alpha}$, such that

$$P^*(\tilde{\beta}) = \tilde{\beta},$$

$$P^* \text{ contracting in } \tilde{\beta}.$$

Let $\|\cdot\|$ denote the euclidean norm in U and let $\beta \in U$. Then

$$P^*(\beta) - \tilde{\alpha} = P^*(\beta) - P^*(\tilde{\alpha}) + P^*(\tilde{\alpha}) - \tilde{\alpha} = P^*(\beta) - P^*(\tilde{\alpha}) + O(\delta^2).$$

By the fact that $D(\alpha+\delta Q(\alpha))$ is contracting in $\tilde{\alpha}$, and that $D^2P^* = O(\delta)$ one has

$$\|P^*(\beta) - P^*(\tilde{\alpha})\| \le (1-\gamma\delta)\|\beta-\tilde{\alpha}\| + M\|\beta-\tilde{\alpha}\|^2 + N\delta^2$$

for some positive γ, M and N. Let $\|\beta-\tilde{\alpha}\| \le 2\delta N/\gamma$ then for δ sufficiently small $\|P^*(\beta)-\tilde{\alpha}\| \le 2\delta N/\gamma$. Thus an $O(\delta)$ ball in U around $\tilde{\alpha}$ is mapped on itself by P^*. It follows that P^* has a fixed point $\tilde{\beta}$ in an $O(\delta)$ neighbourhood of $\tilde{\alpha}$. Using $D^2P^* = O(\delta)$ we see that D is contracting in $\tilde{\beta}$. This concludes the proof of the lemma. □

At this stage our knowledge of the behaviour of the singular solution is sufficient to return to the original equation (3.1). Combining lemma 3.6, lemma 3.4 and theorem 3.2 we obtain:

THEOREM 3.2. *Suppose that $\tilde{\alpha}$ satisfies the three conditions of lemma* 3.6. *Then a positive $\bar{\delta}$ and a positive function $\bar{\varepsilon}(\delta)$, defined on $(0,\bar{\delta})$ exist such that for $0 < \delta < \bar{\delta}$ and $0 < \varepsilon < \bar{\varepsilon}(\delta)$ equation* (3.1) *has a periodic solution. This periodic solution has the following properties:*

(i) *Its trajectory* $Z_{\varepsilon,\delta}$ *tends to the trajectory of* $(x^0(\tilde{\alpha}_1+t),\dots,x^0(\tilde{\alpha}_n+t),\ y^0(\tilde{\alpha}_1+t),\dots,y^0(\tilde{\alpha}_n+t))$ *when* ε *and* δ *tend to zero.*

(ii) *Its period* $P_{\varepsilon,\delta}$ *satisfies* $P_{\varepsilon,\delta} = T_0 - \frac{\delta}{n}\sum G_i(\tilde{\alpha}) + O(\delta^{3/2}) + O(\varepsilon^{2/3})$.

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LITERATURE

[1] CODDINGTON, E.A. & N. LEVINSON, *Theory of ordinary differential equations*, (McGraw-Hill, New York, 1955).

[2] GRASMAN, J. & M.J.W. JANSEN, *Mutually synchronized relaxation oscillations as prototypes of oscillating systems in biology*, (Mathematical Centre, report TW 178/78, Amsterdam 1978).

[3] HALE, J.K., *Ordinary differential equations*, (Wiley-Interscience, New York, 1969).

[4] HIRSCH, M.W. & S. SMALE, *Differential equations, dynamical systems and linear algebra*, (Academic Press, New York, 1974).

[5] HOPPENSTEADT, F., *Properties of solutions of ordinary differential equations with small parameters*, Comm. Pure and Appl. Math., vol. 24 (1971), 807-840.

[6] LASALLE, J., *Relaxation oscillations*, Quart. J. Appl. Math. 7 (1949), 1-19.

[7] MISHCHENKO, E.F., *Asymptotic calculation of periodic solutions of systems of differential equations containing small parameters in the derivatives*, Izv. Akad. Nauk. SSSR, ser. mat., 21 (1957), 627-654, AMS Translations, series 2, vol. 18, 1961, 199-230.

[8] MISHCHENKO, E.F., & L.S. PONTRYAGIN, *Derivation of some asymptotic estimates for solutions of differential equations with a small parameter in the derivative (in Russian)*, Izv. Akad. Nauk. SSR, ser. mat., 23 (1959), 643-660.

[9] MISHCHENKO, E.F. & N.KH. ROZOV, *Differential equations with a small parameter and relaxation oscillations (in Russian)*, (Nauka, Moscow, 1975).

[10] PONTRYAGIN, L.S., *Asymptotic behaviour of solutions of systems of differential equations with a small parameter in the derivatives*, Izv. Akad. Nauk SSSR, ser. mat., 21 (1957), 605-626, AMS Translations, series 2, vol. 18, 1961, 295-320.

[11] ROEVER, J.W. DE (ed.), *Colloquium onderwerpen uit de biomathematica*, Mathematical Centre, Amsterdam, 1976 (vol. 1) and 1977 (vol. 2).

[12] TIKHONOV, A.N., *Systems of differential equations containing small parameters in the derivatives (in Russian)*, Mat. Sbornik, vol. 31 (73), 1952, 575-586.

[13] WINFREE, A., *Biological rhythms and the behaviour of populations of coupled oscillators*, J. Theor. Biol. 16 (1957), 15-42.